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1994 J. Phys. A: Math. Gen. 27 6315

(http://iopscience.iop.org/0305-4470/27/19/007)

J. Phys. A: Math. Gen. 27 (1994) 6315–6334. Printed in the UK

Random walk models for multifractals

Thierry Huillet† and Bernard Jeannet
LIMHP–CNRS, Institute Galilée, Paris 13, Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

Received 25 October 1993, in final form 21 March, 1994

Abstract. Reversible random walk models for *Bernoulli* multifractals in dimension 2 (together with marginal versions of these) are first studied to account for both geometrical and analytical properties of such objects, using large deviation theory. It is also shown how the transience (recurrence) of a more complex reversible random walk is related to the minimum energy flow convergence (divergence) property. New results on the asymptotics of such a walker are then derived in both transient/recurrent situations.

1. Introduction

Multifractal analysis has recently emerged as an important concept in various fields, including strange attractors for dynamical systems, percolating clusters, diffusion limited aggregate growth models ... [3–6, 8, 13, 16, 18]. This formalism has been designed in order to account for the statistical scaling properties of singular measures when it happens that a finite mass can be spread over a region of phase space in such a way that its distribution varies widely. The multifractal formalism hangs upon the definition of the singularity spectrum which associates to the subset of the support of the measure where singularity has given strength, its *Hausdorff* dimension.

The purpose of this paper is first to reconsider this problem in the case of a *Bernoulli* cascade, from the point of view of simple irreversible two-dimensional random walk models to which large-deviation results can be applied and which prove a very useful tool as far as the asymptotics of the distribution is concerned.

Second, we are interested in second-order statistics of joint volume and mass fractions processes, i.e. into the 'energy' of the splitting process. It happens that under certain conditions that we define, energy diverges. Finally, the well known relation [10, 15] of this problem to the recurrence/transience transition of a particular reversible denumberable *Markov* chain is discussed in some detail.

2. Definitions

We define the following problems that we consider.

Fix the (integer) topological dimension d_I of some 'regular' (i.e. non-fractal) compact set I, to be called the initiator. Let V denote its 'volume'. Suppose, now, a mass m, uniformly distributed on I, splits into M sub-masses:

$$m_1 \stackrel{\text{def}}{=} m\pi_1, \ldots, m_M \stackrel{\text{def}}{=} m\pi_M \qquad (\text{with } \pi_l > 0,\ \sum_{l=1}^{M} \pi_l = 1)$$

† E-mail address: huillet@d.univ-paris13.fr

respectively distributed on the sub-volumes of I:

$$V_l \stackrel{\text{def}}{=} Vf_1, \ldots, V_M \stackrel{\text{def}}{=} Vf_M \qquad (f_l \in]0,1[,\ l = 1, \ldots, M).$$

Three different cases arise:

(1) *Contraction.* This happens if $\sum_{l=1}^M f_l < 1$, in which case the fraction $f_0 \stackrel{\text{def}}{=} 1 - \sum_{l=1}^M f_l$ of the volume V receives a zero mass. (This situation is encountered, for example, with the middle third *Cantor* set, with $I = [0, 1]$.)

(2) *Expansion.* This happens if $\sum_{l=1}^M f_l > 1$, in which case the initiator I 'grows' in some embedding space of higher dimension than d_l (this situation is encountered, for example, with the *von Koch* curve with $I = [0, 1]$, and with *Mandelbrot*'s fractal σ-clusters [13]).

(3) *Critical case.* When $\sum_{l=1}^M f_l = 1$.

The above generator defines the first step of the procedure to be indefinitely iterated, since each subvolume splits into sub-subvolumes associated with sub-submasses in the same ratios (f_l, π_l) as for the initiator, ...

Intimately associated with this problem is an *M-Cayley* tree (figure 1), Γ, for which the label $(1, 1)$ is attached to the root, and $(\mu(i), \varphi(i))$ to any other vertex i, at distance $|i|$ from the root, with

$$\mu(i) = \prod_{j=1}^{|i|} \pi_{l_j(i)} \qquad\qquad l_j(i) \in \{1, \ldots, M\},\ j = 1, \ldots, |i|$$
$$\varphi(i) = \prod_{j=1}^{|i|} f_{l_j(i)}$$

the natural decompositions of the considered cylinder $i \in \{1, \ldots, M^{|i|}\}$.

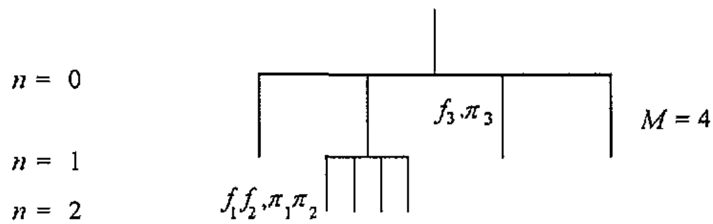

Figure 1. The Cayley tree.

Note that no self-similarity of the chunks is assumed here, and that if this were to be the case, then $f_l \stackrel{\text{def}}{=} (\ell_l)^{d_l}, l = 1, \ldots, M$, with ℓ_l their similarity ratios, would fit the definitions.

As is well known [6, 8, 9], much information on the limit mass distribution is encapsulated in the joint partition function:

$$\Psi_n(\lambda_1, \lambda_2) = \sum_{i=1}^{M^n} \mu(i)^{\lambda_1} \varphi(i)^{\lambda_2} \qquad (\lambda_1, \lambda_2) \in \mathbb{R}^2$$
$$= \left(\sum_{l=1}^{M} \pi_l^{\lambda_1} f_l^{\lambda_2} \right)^n$$

and in the free energy function

$$F(\lambda_1, \lambda_2) = -\frac{1}{n} \log_M \Psi_n(\lambda_1, \lambda_2). \tag{1}$$

Before entering in some detail into these considerations, let us give a stochastic formulation of this problem.

3. The equivalent stochastic formulation

The above enumeration problem has a clear counterpart that we wish to discuss now.

At step n, pick up one chunk $i_{(n)}$ among $\{1, \ldots, N_n \stackrel{\text{def}}{=} M^n\}$ at random. To do this, first note that $i_{(n)}$ can nicely be represented by a random vector $i_{(n)} \stackrel{\text{def}}{=} (L_1, \ldots, L_j, \ldots, L_n)$, with L_j, $j = 1, \ldots, M$, independent and identically distributed random variables equidistributed on $\{1, \ldots, M\}$, i.e. $P(L_j = l) = 1/M$, $l \in \{1, \ldots, M\}$, $j \in \{1, \ldots, n\}$.

The random mass fraction attributed to $i_{(n)}$ is therefore

$$\mu(i_{(n)}) = \prod_{j=1}^{n} \pi_{L_j}$$

and its random volume fraction

$$\varphi(i_{(n)}) = \prod_{j=1}^{n} f_{L_j} .$$

Letting, then, the log variables be

$$\boldsymbol{X}_n \stackrel{\text{def}}{=} \begin{pmatrix} X_{n,1} \\ X_{n,2} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} -\log_M \mu(i_{(n)}) \\ -\log_M \varphi(i_{(n)}) \end{pmatrix} \tag{2}$$

then $\boldsymbol{X}_n$ is easily seen to be a discrete time process with independent increments (PII) with value in $\mathbb{R}^{+2}$ recursively defined by the random walk:

$$\boldsymbol{X}_{n+1} = \boldsymbol{X}_n + \Delta \boldsymbol{X}_{n+1} \qquad \boldsymbol{X}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

the increments' law being given by their real *Laplace* transform, presenting a statistical dependence between the two components:

$$\boldsymbol{E}(M^{\lambda.\Delta X}) = \frac{1}{M} \sum_{l=1}^{M} \pi_l^{\lambda_1} f_l^{\lambda_2}.$$

Consequently

$$\boldsymbol{E}(M^{-\lambda.X_n}) = \left(\frac{1}{M} \sum_{l=1}^{M} \pi_l^{\lambda_1} f_l^{\lambda_2} \right) \stackrel{\text{def}}{=} \frac{1}{M^n} \Psi_n(\lambda).$$

This stochastic formulation of the former enumeration problem allows for the use of the 2D large-deviation theorem and some marginal versions of it.

It is indeed known from the law of large numbers that

$$\lim_{n\to\infty} \frac{1}{n} X_n \overset{\text{a.s.}}{=} \overline{\Delta X}$$

with $\overline{\Delta X}$ the increments' average, and that, from the central limit theorem,

$$\sqrt{n}\,\Sigma^{-1/2}\left(\frac{1}{n} X_n - \overline{\Delta X}\right)$$

has asymptotic normal distribution of zero mean, identity variance, with Σ the variance–covariance matrix of the increments.

Also, one learns from large-deviation theory [1] that, letting the concave analytic cumulant function be:

$$-\log_M(E(M^{-\lambda.\Delta X})) \overset{\text{def}}{=} 1 + F(\lambda) \qquad \lambda \in \mathbb{R}^2$$

and

$$f(\alpha) \overset{\text{def}}{=} \inf_{\lambda}(\alpha \cdot \lambda - F(\lambda)) \tag{3}$$

be the analytic *Legendre* transform of F (figure 2), defined on the convex hull S of the set of points $(-\log_M \pi_l, -\log_M f_l)$, $l = 1, \ldots, M$, then:

Theorem 1. For any convex *Borel* set $A \subset S$, not including $\overline{\Delta X}$:

$$\lim_{n\to\infty} P\left(\frac{1}{n} X_n \in A\right)^{1/n} = M^{\left(\sup_{\alpha\in A} f(\alpha) - 1\right)}. \qquad \square$$

In this theorem, the function $f(\alpha)$ has the following remarkable interpretation (see figure 3):

Proposition 1. Let

$$f(\lambda) \overset{\text{def}}{=} f(\nabla F(\lambda)) = \lambda \cdot \nabla F(\lambda) - F(\lambda) \qquad \lambda \in \mathbb{R}^2$$

be the λ-representation of $f(\alpha)$. Let

$$\rho_l(\lambda) \overset{\text{def}}{=} \frac{\pi_l^{\lambda_1} f_l^{\lambda_2}}{\sum_{l=1}^{M} \pi_l^{\lambda_1} f_l^{\lambda_2}} \qquad l = 1, \ldots, M$$

be a two-parameter family of probability measures, then:

$$f(\lambda) = -\sum_{l=1}^{M} \rho_l(\lambda) \log_M \rho_l(\lambda)$$

i.e. $f(\lambda)$ is the *Shannon* entropy of $\rho(\lambda)$. $\square$

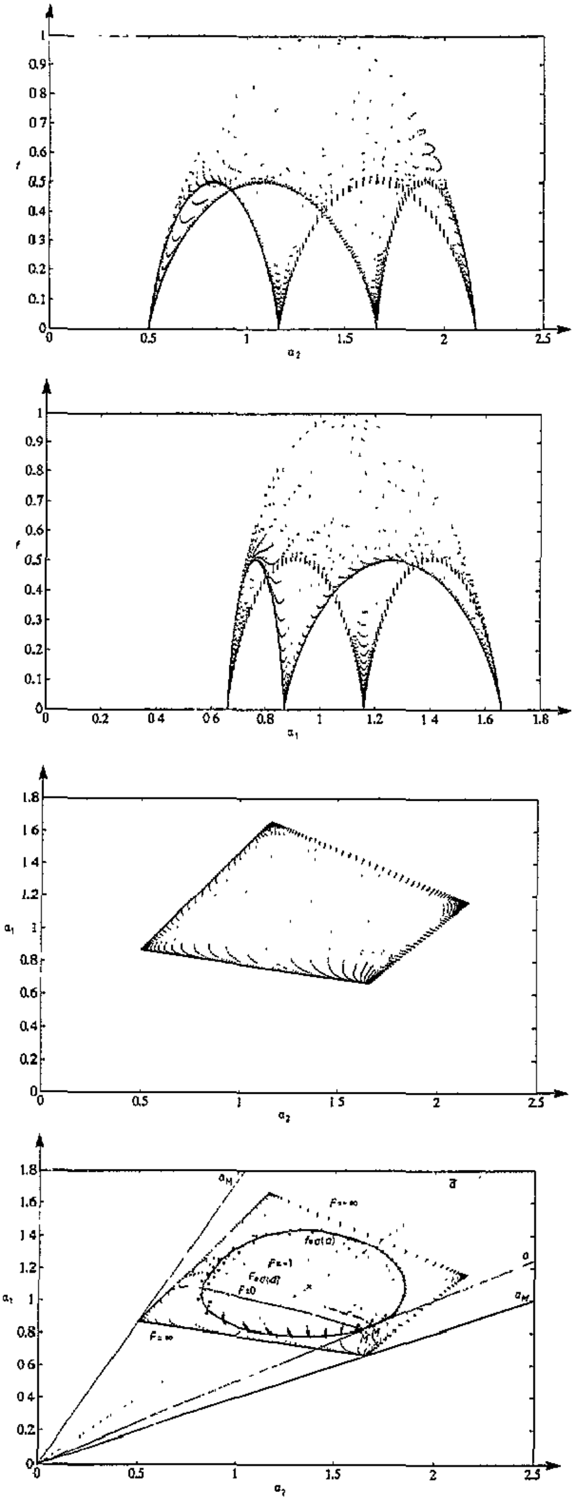

Figure 2. This plot represents the different views of f as a function of α. In the lower right figure the dashed curves within the domain S represent various level lines of F that are of interest.

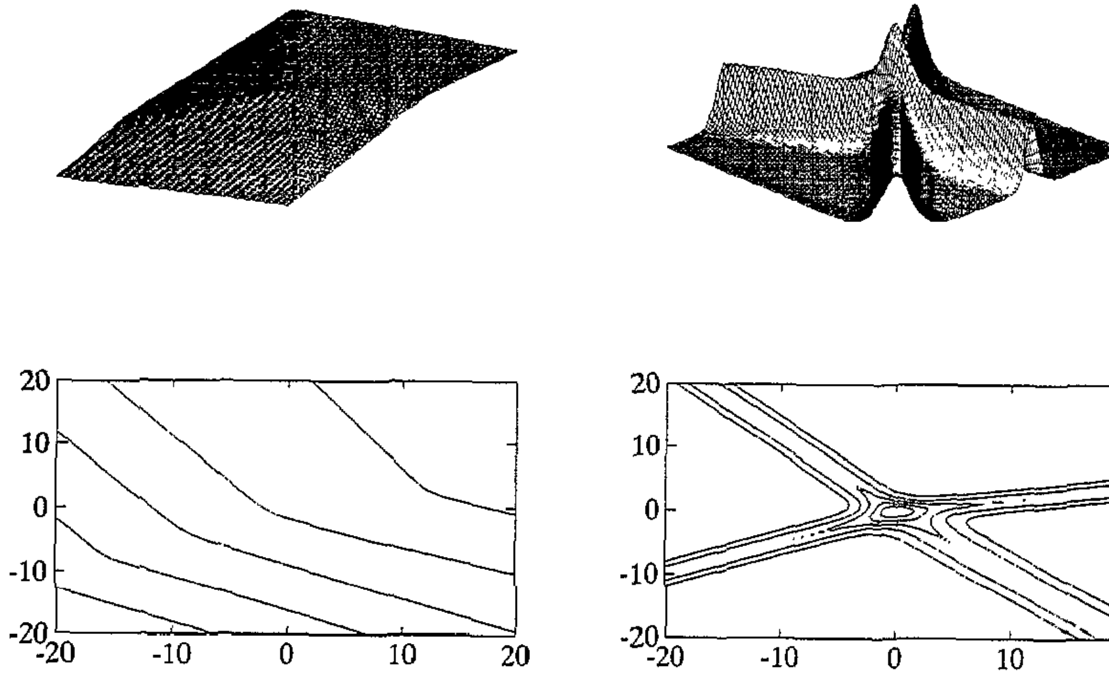

Figure 3. The upper left plot represents the graphs of F, whereas the upper right one is the λ-representation of the Shannon entropy f. Their respective level lines are beneath.

Before introducing now a related problem of interest in ID, let us first define the following quantities:

$$a_m \stackrel{\text{def}}{=} \inf_{l=1,\dots,M} \left(\frac{\log \pi_l}{\log f_l}\right) \qquad a_M \stackrel{\text{def}}{=} \sup_{l=1,\dots,M} \left(\frac{\log \pi_l}{\log f_l}\right) \qquad \bar{a} \stackrel{\text{def}}{=} \frac{\sum_{l=1}^{M} \log \pi_l}{\sum_{l=1}^{M} \log f_l}.$$

Using these notations, the following holds:

Corollary 1. If $a \in]\bar{a}, a_M]$,

$$F_a(\lambda) \stackrel{\text{def}}{=} F(\lambda, -a\lambda) \qquad f_a(\alpha) \stackrel{\text{def}}{=} \inf_{\lambda}(\alpha \cdot \lambda - F_a(\lambda)) \qquad \text{with } \alpha \in [\alpha_m(a), \alpha_M(a)]$$

and

$$\alpha_m(a) \stackrel{\text{def}}{=} \inf_{l=1,\dots,M} (-\log_M(\pi_l f_l^{-a})) < 0$$

and

$$\alpha_M(a) \stackrel{\text{def}}{=} \sup_{l=1,\dots,M} (-\log_M(\pi_l f_l^{-a})) > 0.$$

Then,

$$\lim_{n\to\infty} P\left(\frac{\log \mu(i_{(n)})}{\log \varphi(i_{(n)})} \geqslant a\right)^{1/n} = M^{\sigma(a)-1} \qquad \text{with} \qquad \sigma(a) \stackrel{\text{def}}{=} \sup_{\alpha \geqslant 0} f_a(\alpha) \text{ in } [0, 1[\,. \tag{4}$$

Proof. Let $Y_{n,a} \stackrel{\text{def}}{=} X_{n,1} - aX_{n,2}$, with a a fixed real constant. One wants to evaluate $\boldsymbol{P}(Y_{n,a} \geqslant 0)$ for large n. Notice first that the event $Y_{n,a} \geqslant 0$ also is, from (2):

$$\frac{\log \mu(i_{(n)})}{\log \varphi(i_{(n)})} \geqslant a$$

and that in order that this problem has no trivial solution a must fall in the range $]\bar{a}, a_M]$ (the symmetric event $Y_{n,a} \leqslant 0$ should be considered if $a \in [a_m, \bar{a}[$, in the same way). As a consequence,

$$\boldsymbol{P}(Y_{n,a} \geqslant 0) = \boldsymbol{P}\left(\frac{1}{n} Y_{n,a} \geqslant 0\right)$$

and the above statement is a one-dimensional version of the large-deviation theorem applied to the PII $Y_{n,a}$. □

From the definition of $f_a(\alpha)$ one obtains $f_a(\alpha) = \alpha\lambda_a^*(\alpha) - F_a(\lambda_a^*(\alpha))$ with $F_a'(\lambda_a^*(\alpha)) \stackrel{\text{def}}{=} \alpha$, so that:

$$\sigma(\alpha) = f_a(0) = -F_a(\lambda_a^*(0)) = \log_M \sum_{l=1}^{M} (\pi_l f_l^{-a})^{\lambda_a^*(0)}$$

where $F_a'(\lambda_a^*(0)) = 0$. Figure 4 below represents $\sigma(a)$.

Also, letting $\alpha_a^*(\lambda)$ be the inverse of $\lambda_a^*(\alpha)$, and $f_a(\lambda) \stackrel{\text{def}}{=} \lambda F_a'(\lambda) - F_a(\lambda)$ be the λ-representation of $f_a(\alpha)$, one obtains the well known result [7, 17]:

Proposition 2. Let

$$\rho_{l,a}(\lambda) \stackrel{\text{def}}{=} \frac{(\pi_l f_l^{-a})^\lambda}{\sum_{l=1}^{M} (\pi_l f_l^{-a})^\lambda} \qquad l = 1, \ldots, M$$

be a one-parameter family of probability measure. Then $f_a(\lambda)$ is the *Shannon* entropy of $\rho_{l,a}(\lambda)$.

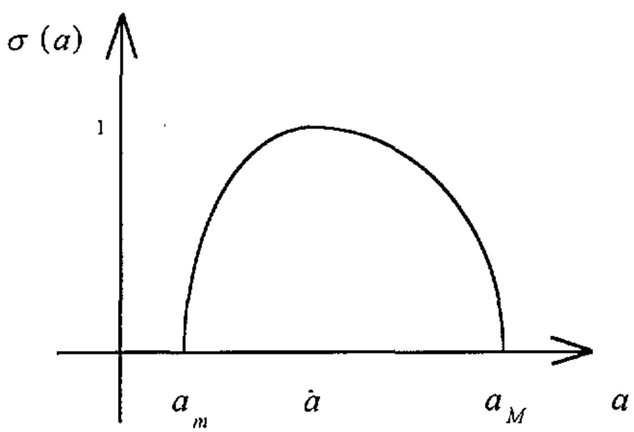

Figure 4. The graph of $\sigma(a)$.

Also, letting

$$D_a(\lambda) \stackrel{\text{def}}{=} \frac{1}{\lambda - 1} F_a(\lambda)$$

be the *Rényi* entropy of $\pi_l f_l^{-a}$, $l = 1, \ldots, M$, and

$$G_a(\lambda) \stackrel{\text{def}}{=} \sum_{l=1}^{M} \rho_{l,a}(\lambda) \log_M \frac{\pi_l f_l^{-a}}{\rho_{l,a}(\lambda)}$$

the *Kullback* information gain, then the following relation holds:

$$f_a(\lambda) = D_a(\lambda) - \frac{\lambda}{\lambda - 1} G_a(\lambda). \qquad \square$$

Remark 1. Another immediate consequence of theorem 1 concerns the asymptotics of the log-density process defined by:

$$Y_{n,1} = X_{n,1} - X_{n,2} \stackrel{\text{def}}{=} -\log \rho(i_{(n)}) \qquad \text{with } \rho(i_{(n)}) \stackrel{\text{def}}{=} \frac{\mu(i_{(n)})}{\varphi(i_{(n)})}$$

the random density of the chunk $i_{(n)}$.

Introducing

$$\rho_l \stackrel{\text{def}}{=} \frac{\pi_l}{f_l} \qquad l = 1, \ldots, M$$

$$\alpha_m(1) \stackrel{\text{def}}{=} \inf_{l=1,\ldots,M} (-\log_M \rho_l) \qquad \alpha_M(1) \stackrel{\text{def}}{=} \sup_{l=1,\ldots,M} (-\log_M \rho_l)$$

and the average density

$$\bar{\alpha}_m(1) \stackrel{\text{def}}{=} -\frac{1}{M} \sum_{l=1}^{M} \log_M \rho_l$$

then for all $\alpha_0 \in]\bar{\alpha}(1), \alpha_M(1)]$:

$$\lim_{n\to\infty} P\left(\frac{1}{n} Y_{n,1} \geqslant \alpha_0\right)^{1/n} = M^{\sup_{\alpha \geqslant \alpha_0} f_l(\alpha) - 1}.$$

Alternatively:

$$\lim_{n\to\infty} \frac{1}{n} \log_M \sharp\{i \in \{1, \ldots, N_n\} \mid \rho(i)^{1/n} \leqslant \rho_0\} = f_l(-\log_M \rho_0)$$

in $[0, 1[$ with $\rho_0 \stackrel{\text{def}}{=} M^{-\alpha_0}$.

4. The singularity spectrum

Let $\tilde{f}_a(\alpha_2) \stackrel{\text{def}}{=} f(a\alpha_2, \alpha_2)$ now denote the intersection of the graph of the function $f(\alpha)$ with the plane having equation $\alpha_1 = a\alpha_2$ (figure 5).

Theorem 2. $\tilde{f}_a(\alpha_2)$ is maximum at $\alpha_2^M(a)$, located at the point M of the plane with the equation $\alpha_1 = a\alpha_2$ for which $\tilde{f}_a(\alpha_2^M(a)) = -F \stackrel{\text{def}}{=} \sigma(a)$.

$\tilde{f}_a(\alpha_2)$ is tangent at $\alpha_2^N(a)$ to a line of slope $g(a)$ passing through the origin, N, for which $\alpha_2 = \alpha_2^N(a)$ is the point of the plane with equation $\alpha_1 = a\alpha_2$ for which $F = 0$ (see figure 5).

Moreover, $g(a) = \inf_{\lambda_1}(a\lambda_1 + \tau(\lambda_1))$, $a \in [a_m, a_M]$, is real concave analytic as the *Legendre* transform of the real convex analytic solution $\tau(\lambda_1)$ of $F(\lambda_1, \tau(\lambda_1)) = 0$. This defines a unique volume fraction scale

$$f^P(a) \stackrel{\text{def}}{=} M^{-\alpha_2^P(a)} \qquad \alpha_2^P(a) \stackrel{\text{def}}{=} \frac{\sigma(a)}{g(a)}.$$

The point P, for which $\alpha_2 = \alpha_2^P(a)$, is located at the intersection of the plane with equation $\alpha_1 = a\alpha_2$ and the level line $F = -x(a)$, with $x(a) < \sigma(a)$ geometrically characterized in figure 5.

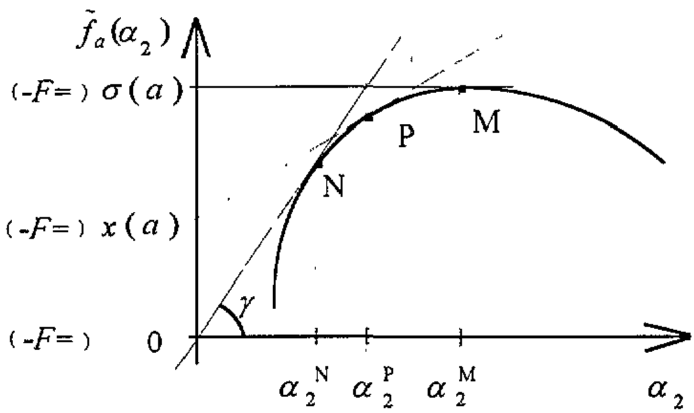

Figure 5. The a-cutset of f and the various pieces of information attached to it.

Proof. From (3), $f(\alpha)$ is also $f(\alpha) = \boldsymbol{\lambda}^*(\alpha) \cdot \alpha - F(\boldsymbol{\lambda}^*(\alpha))$ with $\nabla_\lambda F(\boldsymbol{\lambda}^*(\alpha)) = \alpha$ or $\nabla_\alpha f(\alpha) = \boldsymbol{\lambda}^*(\alpha)$ defining $\boldsymbol{\lambda}^*(\alpha)$. Therefore $\tilde{f}_a'(\alpha_2) = \lambda_1^*(a\alpha_2, \alpha_2)a + \lambda_2^*(a\alpha_2, \alpha_2)$. Defining Q as a point of the curve $\tilde{f}_a(\alpha_2)$ for which $\alpha_2 = \alpha_2^Q(a)$, let us derive the equation of the tangent at Q. It is:

$$y = \tilde{f}_a(\alpha_2^Q(a)) + p^Q(a)(\alpha_2 - \alpha_2^Q(a))$$

with

$$p^Q(a) = \lambda_1^*(a\alpha_2^Q(a), \alpha_2^Q(a))a + \lambda_2^*(a\alpha_2^Q(a), \alpha_2^Q(a)).$$

It is also

$$y = p^Q(a)\alpha_2 - F(\lambda^*(a\alpha_2^Q(a), \alpha_2^Q(a))).$$

It intersects the axis $\alpha_2 = 0$ at $y = -F(\lambda^*(a\alpha_2^Q(a), \alpha_2^Q(a)))$. We focus our attention on three particular points:

(i) $Q = M$. It is the case $p = 0$. M is the maximum of the function $\tilde{f}_a(\alpha_2)$. The two representations of the equation of the tangent allow us to write

$$\tilde{f}_a(\alpha_2^M(a)) = -F(\lambda^*(a\alpha_2 M(a), \alpha_2^M(a))) \stackrel{\text{def}}{=} \sigma(a)$$

i.e.

$$\lambda \cdot \nabla_\lambda F(\lambda) = \alpha \cdot \nabla_\alpha f(\alpha) = 0.$$

(ii) $Q = N$. It is the particular point for which $F = 0$. It is the only point of the curve $\tilde{f}_a(\alpha_2)$ for which the tangent goes through the origin. We define $p^N(a) \stackrel{\text{def}}{=} g(a)$ to be the slope of this curve in N. The result $\tilde{f}_a(\alpha_2^N(a)) = g(a)\alpha_2^N(a)$ follows immediately. The geometric properties of Legendre transforms allows us to conclude as for $g(a) = \inf_{\lambda_1}(a\lambda_1 + \tau(\lambda_1))$, $a \in [a_m, a_M]$.

(iii) $Q = P$. This point is defined thanks to $\alpha_2^P(a) \stackrel{\text{def}}{=} \sigma(a)/g(a)$. In this point, P, the intersection of the tangent with the axis $\alpha_2 = 0$, $x(a) \stackrel{\text{def}}{=} -F(\lambda^*(a\alpha_2^P(a), \alpha_2^P(a)))$ geometrically characterizes the required value of F. □

Remark 2. It should now be noted that the large deviation result (4) can be reformulated in the following equivalent manner:

If $a \in]\bar{a}, a_M]$:

$$\lim_{n\to\infty} \frac{1}{n} \log_M \sharp \left\{ i \in \{1, \ldots, N_n\} \,\middle|\, \frac{\log \mu(i)}{\log \varphi(i)} \geqslant a \right\} = \sigma(a) = g(a)\alpha_2^P(a)$$

or:

$$\lim_{n\to\infty} -\frac{1}{n} \log_{f^{P(a)}} \sharp \left\{ i \in \{1, \ldots, N_n\} \,\middle|\, \frac{\log \mu(i)}{\log \varphi(i)} \geqslant a \right\} = g(a) \tag{5}$$

giving the asymptotics of *Hölder*'s exponents re-partition in the tree.

Let us study in more detail the function $s(a) \stackrel{\text{def}}{=} d_I g(a)$, which is the singularity spectrum of the measure μ [2, 4, 5, 16, 18].

First observe the geometric interpretation of $g(a)$ (see figure 6) is the following. Fix $a \in [a_m, a_M]$; a line of slope $-a$ is tangent to the function $\tau(\lambda)$ at $\lambda_1^*(a)$. It intersects the line with equation $\lambda_1 = 0$ at $g(a)$.

Thus $g(a)$, $a \in [a_m, a_M]$, varies in the range 0 and d_S, where d_S is the unique solution of $F(0, \lambda_2) = 0$.

From (1),

$$F(0, \lambda_2) = -\log_M \sum_{l=1}^{M} f_l^{\lambda_2}$$

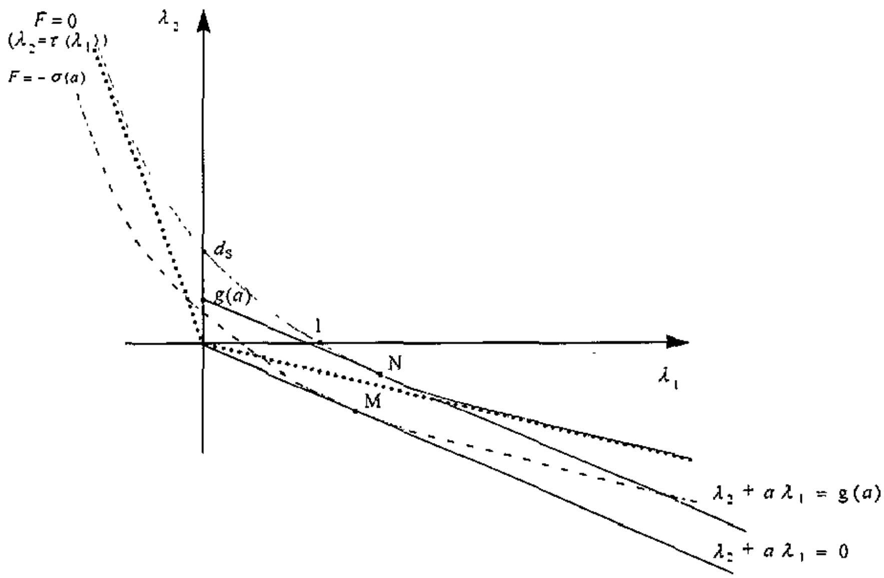

Figure 6. The geometrical interpretation of $g(a)$ in the λ plot.

and thus d_S is the unique solution of:

$$\sum_{l=1}^{M} f_l^{\lambda_2} = 1. \tag{6}$$

Three different cases arise:

$$0 < d_S < 1 \qquad \text{if} \quad \sum_{l=1}^{M} f_l < 1 \qquad \text{(contraction)}$$

$$d_S > 1 \qquad \text{if} \quad \sum_{l=1}^{M} f_l > 1 \qquad \text{(expansion)}$$

$$d_S = 1 \qquad \text{if} \quad \sum_{l=1}^{M} f_l = 1 \qquad \text{(critical case).}$$

In figure 7 the graph of $s(a) \stackrel{\text{def}}{=} \mathrm{d}_I g(a)$ has been represented.

Two remarkable points are worth being discussed.

From theorem 2,

$$g(a) = \inf_{\lambda_1}(a\lambda_1 + \tau(\lambda_1)) = a\lambda_1^*(a) + \tau(\lambda_1^*(a)) \qquad \text{with} \quad \dot{\tau}(\lambda_1^*(a)) = -a.$$

But from the definition of τ, $F(\lambda_1, \tau(\lambda_1)) = 0$, thus

$$\dot{\tau}(\lambda_1) = -\frac{F_1'(\lambda_1, \tau(\lambda_1))}{F_2'(\lambda_1, \tau(\lambda_1))}.$$

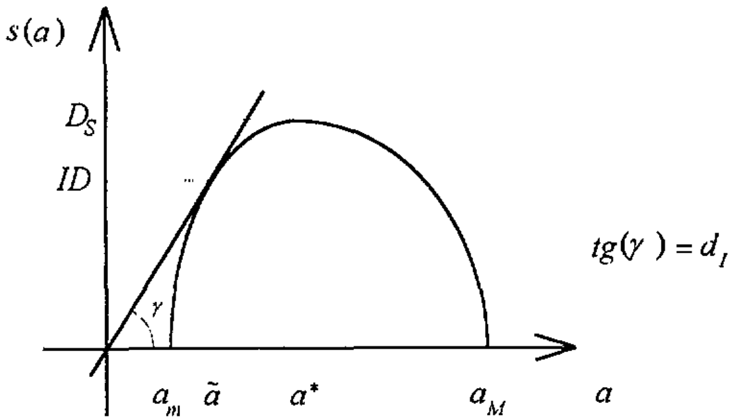

Figure 7. The graph of $s(a)$.

Hence $s(a)$ attains its maximum

$$D_s \stackrel{\text{def}}{=} d_I d_S = d_I \tau(0) \qquad \text{at} \qquad -\dot{\tau}(0) \stackrel{\text{def}}{=} a^* = \frac{\sum_{l=1}^{M} \log \pi_l f_l^{d_S}}{\sum_{l=1}^{M} \log f_l f_l^{d_S}}.$$

D_S is the *Hausdorff* dimension of the concentration set of the procedure defined in section 2 or the dimension or the support of the measure (see [2, 18] for rigourous results).

Also, at $a = \tilde{a}$,

$$\lambda_1^*(a) = 1 \qquad -\dot{\tau}(1) \stackrel{\text{def}}{=} \tilde{a} = \frac{\sum_{l=1}^{M} \pi_l \log \pi_l}{\sum_{l=1}^{M} \pi_l \log f_l} \qquad \text{and} \qquad s(\tilde{a}) = d_I \tilde{a}.$$

$ID(\pi, f) \stackrel{\text{def}}{=} d_I \tilde{a}$ is called the information dimension of the measure [18]. It has the following interpretation. Consider the set of chunks for which $\varphi(i_{(n)}) \approx f^P(a)^n$, for large n, *and* whose coarse *Hölder* exponent is a.

The mass carried by those chunks, from (5), therefore grows like:

$$\mu(a) \approx f^P(a)^{-ng(a)} f^P(a)^{na} = f^P(a)^{n(a-g(a))}$$

(the product of their number by their mass).

Thus, those chunks for which $a = -\dot{\tau}(1)$ carry all the mass, i.e. $\mu(-\dot{\tau}(1)) = 1$. $ID(\pi, f)$ is thus the information dimension of the measure.

Other entities can be defined, namely, the *Rényi* dimension function of the concentration set:

$$R_{\lambda_1} \stackrel{\text{def}}{=} \frac{1}{1-\lambda_1} \tau(\lambda_1)$$

so that:

$$ID(\pi, f) = \lim_{\lambda_a \downarrow 1} R_{\lambda_1} = -d_I \dot{\tau}(1)$$

and the metric *Rényi* entropy of π:

$$D_{\lambda_1} \stackrel{\text{def}}{=} \frac{1}{1-\lambda_1} F(\lambda_1, 0).$$

Remark 3. Once $f_l \in]0, 1[$, $l = 1, \ldots, M$, are given, d_S is totally determined from (6). Thus d_S can be plotted as a function of the vector f. From the implicit function theorem, $d_S(f)$ is analytic on the open cube $]0, 1[^M$, and undefined at the edges $(0, \ldots, 0, 1, 0, \ldots, 0)$, $l = 1, \ldots, M$, of this cube. In the case $M = 2$, it has a 'beak' shape (see figure 8).

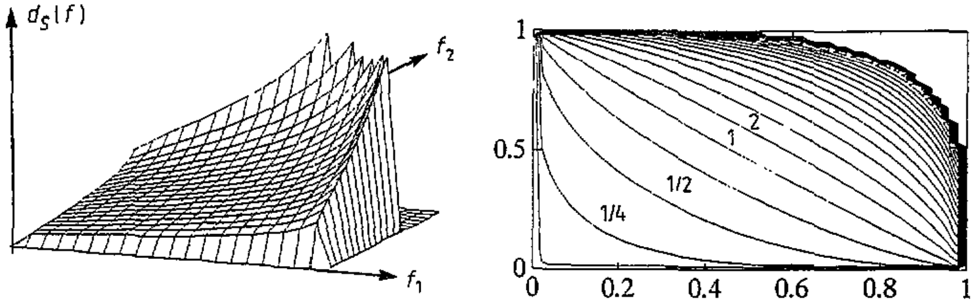

Figure 8. The beak shape surface.

5. Energy

We now come to the study of the energy line $\lambda_1 + \lambda_2 = 2$ of the plane (λ_1, λ_2). Let

$$r_l \stackrel{\text{def}}{=} \frac{1}{\rho_l} = \frac{f_l}{\pi_l} \qquad l = 1, \ldots, M \quad r = (r_1, \ldots, r_M)^t.$$

Define the quantity:

$$E_\Gamma^{r,\pi}(\lambda_2) \stackrel{\text{def}}{=} \sum_{n \geqslant 1} \Psi_n(2 - \lambda_2, \lambda_2) = \sum_{n \geqslant 1} \left(\sum_{l=1}^{M} r_l^{\lambda_2} \pi_l^2 \right)^n \tag{7}$$

to be called the 'energy' of the tree Γ (because second-order statistical properties of the joint mass and volume fraction processes M^{-X_n} are involved).

From (7), this is also:

$$E_\Gamma^{r,\pi}(\lambda_2) = \begin{cases} \dfrac{1}{1 - \sum_{l=1}^{M} r_l^{\lambda_2} \pi_l^2} - 1 & \text{provided } \sum_{l=1}^{M} r_l^{\lambda_2} \pi_l^2 < 1 \\ \infty & \text{otherwise.} \end{cases}$$

Among all flows π, there is one, say $\pi^r(\lambda_2)$, of minimum energy $E_\Gamma^r(\lambda_2) = \text{Argmin}_\pi E_\Gamma^{r,\pi}(\lambda_2) = E_\Gamma^{r,\pi^r(\lambda_2)}(\lambda_2)$ which is shown to be:

$$\pi_l^r(\lambda_2) = \frac{r_l^{-\lambda_2}}{\sum_{l=1}^{M} r_l^{-\lambda_2}} \qquad l = 1, \ldots, M.$$

Thus the energy of the minimum energy flow is:

$$E_\Gamma^r(\lambda_2) = \frac{1}{1 - \sum_{l=1}^{M} r_l^{\lambda_2} \pi_l^r(\lambda_2)^2} - 1 = \frac{1}{1 - \frac{1}{\sum_{l=1}^{M} r_l^{-\lambda_2}}} - 1.$$

This quantity is finite provided the following condition is satisfied:

$$\sum_{l=1}^{M} r_l^{-\lambda_2} = \sum_{l=1}^{M} \rho_l^{\lambda_2} > 1. \tag{8}$$

Remark 4. There is an analogy between this formulation and the one currently used in electrical network theory [10–12, 15], the flow $\mu(i)$ being conserved at each node i of the tree. Thus $r(i)^{\lambda_2}$, $\rho(i)^{\lambda_2}$ can be interpreted as resistances and conductances, respectively.

Also, it is easy to check that on letting $\rho_m = \inf_l \rho_l$ and $\rho_M = \sup_l \rho_l$ three different situations arise:

(1) $\rho_m \leqslant \rho_M < 1$ iff $a_m > 1$, and $\sum_{l=1}^{M} \rho_l^{\lambda_2} = 1$ has a unique positive solution $\lambda_2^+(\rho)$; thus criterion (8) is satisfied when $\lambda_2 < \lambda_2^+(\rho)$.

It should also be noted that if $\sum_{l=1}^{M} f_l > 1$ there is at least an $l \in \{1, \ldots, M\}$ for which $\pi_l < f_l$, so one may conclude that $\rho_m < 1$. Thus, we call the situation under consideration here, $\rho_m \leqslant \rho_M < 1$, the 'strong expansion' case, because the density of the elementary chunks is very low.

(2) $1 < \rho_m \leqslant \rho_M$ iff $a_M < 1$, and $\sum_{l=1}^{M} \rho_l^{\lambda_2} = 1$ has a unique negative solution $\lambda_2^-(\rho)$; thus criterion (8) is satisfied when $\lambda_2 > \lambda_2^-(\rho)$.

We call this situation the 'strong contraction' case, because the density appears very large for symmetrical reasons.

(3) $\rho_m \leqslant 1 \leqslant \rho_M$ iff $a_m < 1 < a_M$, $\sum_{l=1}^{M} \rho_l^{\lambda_2} = 1$ has no solution in λ_2, and criterion (8) is always satisfied. (This situation includes the critical case).

6. Reversible random walk and the energy problem

So far, we have been interested in random walks on the directed tree Γ. Reversible random walks on the undirected tree whose edges are labelled with their conductances (which allows for the definition of transition probabilities) are in natural correspondence with the previous formulation. We wish to work out this analogy now and exhibit some of the results obtained.

We thus define the following random walk on Γ.

A walker starts at the root of the tree and moves downwards on one of the available edges with probability

$$P_l^* \stackrel{\text{def}}{=} \frac{\rho_l^{\lambda_2}}{\sum_{l=1}^{M} \rho_l^{\lambda_2}} \qquad l = 1, \ldots, M.$$

Once any vertex other than the root has been reached, it moves downward with probability

$$P_l \stackrel{\text{def}}{=} \frac{\rho_l^{\lambda_2}}{1 + \sum_{l=1}^{M} \rho_l^{\lambda_2}} \qquad l = 1, \ldots, M$$

and upwards with the remaining probability:

$$P_0 = 1 - \sum_{l=1}^{M} P_l = \frac{1}{1 + \sum_{l=1}^{M} \rho_l^{\lambda_2}} \qquad l = 1, \ldots, M$$

so that:

$$P_l^* = \frac{P_0}{1 - P_0} \rho_l^{\lambda_2} \qquad P_l = P_0 \rho_l^{\lambda_2} \qquad l = 1, \ldots, M.$$

This walker will either return to the root infinitely often almost surely (recurrence) or move to the boundary of the tree (transience).

The associated state we are interested in is $-\log u(i)$, with $u(i) \stackrel{\text{def}}{=} \mu(i)/\rho(i)^{\lambda_2}$ the 'potential' of edge i currently visited by the walker. More precisely, let us now construct recursively the denumerable *Markov* chain of interest:

$$\boldsymbol{X}_{n+1} = \Delta \boldsymbol{X}_1 \cdot \mathbf{1}_{\{X_n^0=0\}} + (\boldsymbol{X}_n + \Delta \boldsymbol{X}_{n+1}) \cdot \mathbf{1}_{\{X_n^0 \leqslant 1\}} \qquad \boldsymbol{X}_0 = (0, \ldots, 0, \ldots)^t \tag{9}$$

Here

$$\boldsymbol{X}_n \stackrel{\text{def}}{=} (|i|_n, -\log u_{l_{|i|_n}}, \ldots, -\log u_{l_1}, 0, \ldots)^t$$

with

$$X_n^0 \stackrel{\text{def}}{=} |i|_n$$

the (random) distance to the root of the currently visited edge i at step n, and $u_{l_j} \stackrel{\text{def}}{=} \pi_{l_j}/\rho_{l_j}^{\lambda_2}$, the local potential variation along the vertex $l_j \in \{1, \ldots, M\}$, $j \in \{1, \ldots, |i|_n\}$. The probability distribution of $\Delta \boldsymbol{X}_1$ is therefore

$$\sum_{l=1}^{M} P_l^* \delta_{\nu_l^*} \qquad \text{with} \qquad \nu_l^* \stackrel{\text{def}}{=} (1, -\log u_l, 0, \ldots)^t$$

(with δ the *Dirac* measure).

The probability distribution of $\Delta \boldsymbol{X}_{n+1}$ is $P_0 \delta_{\nu_0} + \sum_{l=1}^{M} P_l \delta_{\nu_l}$, with

$$\nu_0 \stackrel{\text{def}}{=} A_0 \boldsymbol{X}_n + a_0 \qquad A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ 0 & -1 & 1 & 0 & \ldots \\ 0 & 0 & -1 & 1 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \qquad \text{the upward shift operator}$$

with $a_0 = (-1, 0, 0, \ldots)^t$ the upward shift operator, and

$$\nu_l \stackrel{\text{def}}{=} A_0^t \boldsymbol{X}_n + a_l$$

with $a_l = (1, -\log u_l, 0, \ldots)^t$ defining the downward shift.

6.1. Random distance of the walker to the root

Using *Bayes*' formula, it is easy to derive the result:

Lemma 1. Letting $\boldsymbol{P}(n, q) \stackrel{\text{def}}{=} \boldsymbol{P}(X_n^0 = q)$, for any $q \in \mathbb{N}$, and $\Phi_n(\gamma) = \boldsymbol{E}\, \mathrm{e}^{-\gamma \cdot \boldsymbol{X}_n}$ (with $\gamma \stackrel{\text{def}}{=} (\gamma_0, \gamma_1, \gamma_2, \ldots)^t$), the real *Laplace* transform of $\boldsymbol{X}_n$'s law, the following recurrence holds:

$$\Phi_{n+1}(\gamma) = \mathrm{e}^{-\gamma_0} \sum_{l=1}^{M} P_l^* u_l^{\gamma_1} \boldsymbol{P}(n, 0) + P_0\, \mathrm{e}^{\gamma_0} \left\{ \Phi_n([I + A_0^t]\gamma) - \boldsymbol{P}(n, 0) \right\}$$

$$+ \mathrm{e}^{-\gamma_0} \sum_{l=1}^{M} P_l u_l^{\gamma_1} \{ \Phi_n([I + A_0]\gamma) - \boldsymbol{P}(n, 0) \}$$

with $\boldsymbol{P}(0, 0) = 1$, $\Phi_0(\gamma) = 1$. $\square$

It is of importance (for examining the transition recurrence/transience) to determine the whole distribution of X_n^0, i.e. of the random distance to the root of the walker at time n. A first step in this direction follows immediately from the lemma.

Corollary 2. If $\Phi_n(\gamma_0) \stackrel{\text{def}}{=} \Phi_n(\gamma_0, 0, \ldots) = \boldsymbol{E}\, \mathrm{e}^{-\gamma_0 X_n^0}$ denote the γ_0-marginal, one obtains:

$$\Phi_{n+1}(\gamma_0) = \mathrm{e}^{-\gamma_0}\boldsymbol{P}(n,0) + (\Phi_n(\gamma_0) - \boldsymbol{P}(n,0))(P_0\,\mathrm{e}^{\gamma_0} + (1-P_0)\,\mathrm{e}^{-\gamma_0}) \tag{10}$$

with $\Phi_0(\gamma_0) = 1$. □

Also, it follows from elementary combinatorics that:

$$\boldsymbol{P}(2p+1,0) = 0 \qquad \text{for all } p \geqslant 0$$

$$\boldsymbol{P}(2(p+1),0) = \boldsymbol{P}(2p,0) - \frac{1}{P_0}[P_0(1-P_0)]^{p+1}A_{p+1} \qquad \text{for all } p \geqslant 1 \tag{11}$$

$$\boldsymbol{P}(2,0) = P_0$$

with

$$A_{p+1} = \frac{1}{p+1}\binom{2p}{p}$$

the *Catalan* number.

Putting together all these preliminary results, one obtains:

Theorem 3. The walk defined by (9) (of period 2) is:

(1) transient iff $P_0 < \frac{1}{2}$ (i.e. when condition (8) is satisfied)

(2) recurrent null iff $P_0 = \frac{1}{2}$

(3) recurrent positive iff $P_0 > \frac{1}{2}$.

In this last situatioon,

$$\lim_{p\to\infty} \boldsymbol{P}(2p,0) \stackrel{\text{def}}{=} \boldsymbol{P}(\infty,0) = \frac{2P_0 - 1}{P_0}$$

and:

$$\lim_{p\to\infty} \Phi_{2p+1}(\gamma_0) \stackrel{\text{def}}{=} \Phi_\infty^{\text{odd}}(\gamma_0)$$

$$= \frac{(2P_0-1)(\mathrm{e}^{-\gamma_0} - \mathrm{e}^{\gamma_0})}{1 - ((1-P_0)\,\mathrm{e}^{-\gamma_0} + P_0\,\mathrm{e}^{\gamma_0})^2} = \frac{(2P_0-1)\,\mathrm{e}^{-\gamma_0}}{P_0^2 - (1-P_0)^2\,\mathrm{e}^{-2\gamma_0}}$$

$$\lim_{p\to\infty} \Phi_{2(p+1)}(\gamma_0) \stackrel{\text{def}}{=} \Phi_\infty^{\text{even}}(\gamma_0) = ((1-P_0)\,\mathrm{e}^{-\gamma_0} + P_0\,\mathrm{e}^{\gamma_0})\Phi_\infty^{\text{odd}}(\gamma_0)$$

provided $\mathrm{e}^{-\gamma_0} < P_0/(1-P_0)$, giving the asymptotic distribution for the random distance to the root of the walker. Also:

$$\lim_{p\to\infty} \boldsymbol{P}(p,q) \stackrel{\text{def}}{=} \boldsymbol{P}(\infty,q) = \frac{(1-P_0)^{q-1}}{P_0^q}\boldsymbol{P}(\infty,0) \qquad \text{for any } q \in \mathbb{N}^*. \tag{12}$$

Proof. If $\psi(t) \stackrel{\text{def}}{=} \sum_{p\geqslant 1} \boldsymbol{P}(2p,0)t^{2p}$ denotes the generating function of the series $\boldsymbol{P}(2p,0)$, $p \geqslant 1$, it follows from expression (11) that:

$$\psi(t) \stackrel{\text{def}}{=} \frac{1}{2P_0(1-t^2)}\left[(2P_0t^2-1)+\sqrt{1-4P_0(1-P_0)t^2}\right]$$

provided

$$|t| \leqslant \frac{1}{2\sqrt{P_0(1-P_0)}}\,.$$

The limit $\lim_{t\downarrow 1}\psi(t)$ exists and is $P_0/(1-2P_0)$, provided that $P_0 < \frac{1}{2}$ (transient case). In the case $P_0 \geqslant \frac{1}{2}$ the series $\sum_{p\geqslant 1} \boldsymbol{P}(2p,0)$ diverges (the recurrent case), and from (11) and the well known expression of the generating function for *Catalan*'s numbers:

$$C_t \stackrel{\text{def}}{=} t + \sum_{k\geqslant 2} A_k t^k = \tfrac{1}{2}(1-\sqrt{1-4t}\,)$$

it follows that:

$$\boldsymbol{P}(\infty,0) = P_0 - \frac{1}{P_0}\left(C_{P_0(1-P_0)} - P_0(1-P_0)\right) = P_0 - \frac{1}{P_0}(\tfrac{1}{2}(1-|1-2P_0|) - P_0(1-P_0))\,.$$

Thus:

(i) in the case $P_0 \leqslant \frac{1}{2}$, $\quad \boldsymbol{P}(\infty,0) = 0 \quad$ ($P_0 = \frac{1}{2}$ is a case of null recurrence)

(ii) in the case $P_0 > \frac{1}{2}$, $\quad \boldsymbol{P}(\infty,0) = \dfrac{2P_0-1}{P_0} \quad$ (positive recurrence)

and the expectation of the first return time to zero T_0 is $\boldsymbol{E}(T_0) = 2P_0/(2P_0-1)$, therefore finite.

The limit expressions ending theorem 3 are immediately derived from the fixed point of (10). □

Remark 5. The results relating condition (8) to the transience of some reversible random walk have been known for a long time (see [12, 15] and [10, 11] for a much more general setting of the tree structure). Limiting ourselves to *Cayley* trees allows for complete results.

6.2. Random potential of the walker

Another marginal of $\Phi_n(\gamma)$ that is of interest is $\Phi_n(\gamma_0,\gamma) \stackrel{\text{def}}{=} \Phi_n(\gamma \stackrel{\text{def}}{=} \gamma_0, \gamma, \gamma, \ldots, \gamma_k = \gamma, \ldots))$, giving the joint probability distribution of

$$\left(X_n^0, \sum_{j\geqslant 1} X_n^j \stackrel{\text{def}}{=} -\sum_{j=1}^{X_n^0} \log U_{u_{i_j}} \stackrel{\text{def}}{=} -\log u(i_n)\right)$$

i.e. of the joint distance to the root and 'log potential' states.

From lemma 1, it appears that a recursive equation for $\Phi_n(\gamma_0,\gamma)$ is not so easy to derive, because $\Phi_n([I+A_0^t]\gamma) = \Phi_n(\gamma = \gamma_0, 0, \gamma, \gamma, \ldots)$ is a new entity to generate. Iterating nevertheless, this generation ends up after a finite number of steps on $\Phi_n(\gamma =$

$\gamma_0, 0, 0, \ldots) \stackrel{\text{def}}{=} \Phi_n(\gamma_0)$, whose generation is given in corollary 2. After some tedious computations which we omit here, one can finally show that $\Phi_n(\gamma_0, \gamma)$ is also recursively defined, and:

$$\Phi_{n+1}(\gamma_0, \gamma) = A(\gamma_0, \gamma)\Phi_n(\gamma_0, \gamma) + B_n(\gamma_0, \gamma)$$

with

$$A(\gamma_0, \gamma) = P_0\, \mathrm{e}^{\gamma_0} a^*(\gamma)^{-1} + (1 - P_0)\, \mathrm{e}^{-\gamma_0} a^*(\gamma)$$

$$B_N(\gamma_0, \gamma) = P_0 P(n, 0)(\mathrm{e}^{-\gamma_0} a^*(\gamma) - \mathrm{e}^{\gamma_0} a^*(\gamma)^{-1})$$

$$a^*(\gamma) = \sum_{l=1}^{M} P_l^* u_l^\gamma .$$

This recurrence is defined for $n \geqslant 1$, with $\Phi_0(\gamma_0, \gamma) = 1$, $\Phi_1(\gamma_0, \gamma) = \mathrm{e}^{-\gamma_0} a^*(\gamma)$ defining the initial conditions.

6.2.1. The recurrent case. Therefore in the $P_0 > \frac{1}{2}$ case, since $P(\infty, 0) = (2P_0 - 1)/P_0$:

$$\lim_{p\to\infty} \Phi_{2p+1}(\gamma_0, \gamma) \stackrel{\text{def}}{=} \Phi_\infty^{\text{odd}}(\gamma_0, \gamma)$$

$$= \frac{(2P_0 - 1)B(\gamma_0, \gamma)}{1 - A^2(\gamma_0, \gamma)} = \frac{(2P_0 - 1)a^*(\gamma)\, \mathrm{e}^{-\gamma_0}}{P_0^2 - (1 - P_0)^2 a^*(\gamma)^2\, \mathrm{e}^{-2\gamma_0}}$$

$$\lim_{p\to\infty} \Phi_{2(p+1)}(\gamma_0, \gamma) \stackrel{\text{def}}{=} \Phi_\infty^{\text{even}}(\gamma_0, \gamma)$$

$$= \frac{(2P_0 - 1)A(\gamma_0, \gamma)B(\gamma_0, \gamma)}{1 - A^2(\gamma_0, \gamma)} = A(\gamma_0, \gamma)\Phi_\infty^{\text{odd}}(\gamma_0, \gamma)$$

provided that $a^*(\gamma)\, \mathrm{e}^{-\gamma_0} < P_0/(1 - P_0)$, giving the asymptotic distributions of the joint states, through their *Laplace* transforms.

In particular, $\Phi_\infty^{\text{odd}}(0, \gamma) \stackrel{\text{def}}{=} \boldsymbol{E}u(i_\infty^{\text{odd}})^\gamma$ is defined, provided $a^*(\gamma) < P_0/(1 - P_0)$, i.e. provided $\gamma < 1$ (from the definition of $a^*(\gamma)$), so that the first, $\Phi_\infty^{\text{odd}}(0, 1) \stackrel{\text{def}}{=} \boldsymbol{E}u(i_\infty^{\text{odd}})$, and second order moments of the potential diverge, whereas all limit moments of the random variable $-\log u(i_\infty^{\text{odd}})$ exist as successive derivatives of $\Phi_\infty^{\text{odd}}(0, \gamma) \stackrel{\text{def}}{=} \boldsymbol{E}u(i_\infty^{\text{odd}})$ at $\gamma = 0$.

We have thus related the divergence of the energy of the minimum energy flow to the divergence of the equilibrium second order moment for the potential in the reversible current walk.

The final problem we would like to introduce is the following.

6.2.2. The transient case. Assume the reversible walk is transient. We want to attribute a 'cost function' to any particular trajectory of such a walker and compute the average of this cost over all possible trajectories. More precisely, the cost of the transition $\Delta C_{n,n+1}(\gamma)$ between times n and $n + 1$ is given by:

(i) $u(i_{n+1})^\gamma$ for a downward move of the walker;

(ii) $-u(i_n)^\gamma$ for an upward move of the walker, which means that the walker *restores* his 'potential energy' to the system. In more precise terms:

$$\Delta C_{n,n+1}(\gamma) = -\,\mathrm{e}^{-(0,\gamma,\gamma,\ldots)X_n} \cdot \mathbf{1}_{\{X_{n+1}^0 - X_n^0 = -1\}} + \mathrm{e}^{-(0,\gamma,\gamma,\ldots)X_{n+1}} \cdot \mathbf{1}_{\{X_{n+1}^0 - X_n^0 = 1\}} .$$

Using *Bayes'* formula, one can show that the average cost us such a transition is:

$$E\Delta C_{n,n+1}(\gamma) = (-P_0 + a^*(\gamma)(1 - P_0))(\Phi_n(0, \gamma) - P(n, 0)) + P(n, 0)a^*(\gamma)$$

so that the global average cost over all possible such trajectories is:

$$\overline{C}(\gamma) \stackrel{\text{def}}{=} \sum_{n\geqslant 0} E\Delta C_{n,n+1}(\gamma) = \frac{a^*(\gamma)}{1 - a^*(\gamma)}$$

using the fact that $\sum_{p\geqslant 1} P(2p, 0) = P_0/(1-2P_0)$, in the transient case, and the obvious fact that $\sum_{n\geqslant 0} \Phi_n(0, \gamma)$ converges as a consequence of transience, and after some computations which we omit. Two particular cases are worthy of interest, namely:

$$\overline{C}(1) = \frac{a^*(1)}{1 - a^*(1)} = \frac{P_0}{1 - 2P_0}$$

$$\overline{C}(2) = \frac{a^*(2)}{1 - a^*(2)} = \frac{\dfrac{P_0}{1 - P_0} \displaystyle\sum_{l=1}^{M} r_l^{\lambda_2} \pi_l^2}{1 - \dfrac{P_0}{1 - P_0} \displaystyle\sum_{l=1}^{M} r_l^{\lambda_2} \pi_l^2} = \sum_{n\geqslant 1} \left(\sum_{l=1}^{M} \left(\frac{P_0}{1 - P_0} u_l \right) \pi_l \right)^n .$$

Observing that the minimum energy flow is obtained for $\pi_l = \pi_l^r(\lambda_2) = P_l^*$, then $P_0/(1-P_0)$ is the local potential $u_l \stackrel{\text{def}}{=} r_l^{\lambda_2}\pi_l$ associated to this flow. Therefore, a way to realize this flow is to force the local potential increment between any two connected edges of the tree to be identical and equal to this value, so that, globally,

$$\sum_{n\geqslant 1} \left(\frac{P_0}{1 - P_0} \right)^n = \frac{P_0}{1 - 2P_0}$$

is the potential variation between the root and any vertex of the boundary of the tree.

If this constraint is not satisfied, we observe that the reversible walker measure, through $\overline{C}(2)$, the energy dissipated by the tree as defined by the local current and potential increments $\pi_l = u_l/r_l^{\lambda_2}$ and $(P_0/(1 - P_0))u_l$, respectively.

We have thus associated a 'cost functional' to the reversible transient walker, the finiteness of which in the average is to be related to the existence of the energy of the minimum energy flow.

7. Concluding remarks

This paper has been concerned with random walk models for multifractals where both mass and space have been partitioned.

The first part has looked at the formalism of a *Bernoulli* cascade in terms of a two-dimensional process with independent increments. This has allowed for a direct use of the asymptotic result (theorem 1 concerning the deviation to the central limit theorem, the rate of growth of the deviation probability being related to the *Shannon* entropy f of a two-parameter exponential family of probability measures (proposition 1).

The geometry of the two-dimensional problem has also been underlined. Considering then the related one-dimensional process, $Y_{n,a}$, allowed for the large-deviation results (corollary 1), concerning the ratio process, log(mass)/log(volume) of the atoms, and underlined the essential role of the quantity $\sigma(a)$, the geometrical interpretation of which has been given. More work is probably necessary here to fully understand this quantity (in particular its *Legendre* transform).

Finally, considering a cutset $\bar{f}_a$ of f has allowed for a geometrical characterization of $g(a)$ (and hence $s(a)$), which is the standard singularity spectrum of the multifractal measure, as originally introduced in [6], and extensively studied by *Mandelbrot* and coworkers in the sequel. In particular, its geometrical relationship to $\sigma(a)$ has been underlined.

The second part of the paper has been devoted to the 'energy' of the splitting process, i.e. to second-order statistics of joint volume and mass fraction processes, and the possibility of the divergence (*resp.* convergence) of the minimum energy flow. Although results relating this possibility to the recurrence (*resp.* transience) of some reversible random walk on the tree are already available (see [12, 15] and [10, 11] for a much more general setting of the tree structure), we have shown how complete results (theorem 3) can be obtained if one limits oneself to the case of the *Bernoulli* cascades. In particular, the limit distributions of both random distance to the root and potential of the reversible walker have been explicitly computed in the case of recurrence positiveness.

Finally, we have related the divergence of the energy to the divergence of the order γ (> 1) fractional moment for the limit potential of the recurrent walker. In a dual way, we have shown that the energy convergence is to be related to the finiteness of a certain cost-functional of interest attached to the transient walker.

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